Theoretische Physik 2: Elektrodynamik
(Prof. A.-S. Smith)

Home assignment 3

Problem 3.1  Dirac delta function

Using Dirac delta function in the appropriate coordinates, express the following charge distributions as three-dimensional charge densities $\rho_e(\vec{x})$.

a) In spherical coordinates, a charge $Q$ uniformly distributed over a spherical shell of radius $R$.

b) In cylindrical coordinates, a charge $\lambda$ per unit length uniformly distributed over a cylindrical surface of radius $b$.

c) In cylindrical coordinates, a charge $Q$ spread uniformly over a flat circular disc of negligible thickness and radius $R$.

d) The same as part (c), but using spherical coordinates.

Problem 3.2  Properties of the Dirac delta function

The function $h(x)$ has only one simple root $x_0$. Explain the relation

$$\delta(h(x)) = \frac{1}{|h'(x_0)|}\delta(x - x_0).$$

Prove the following properties of the $\delta$-function:

a) $x\delta(x) = 0$

b) $\varphi(x)\delta(x - a) = \varphi(a)\delta(x - a)$

c) $\int dy \delta(x - y)\delta(y - z) = \delta(x - z)$
Problem 3.3  Legendre polynomials and spherical harmonics

You have been told about the mathematic properties of Legendre polynomials [spherical harmonics] as angular solutions of the Laplace equation in spherical coordinates in the presence [absence] of circular symmetry. To obtain an intuitive understanding of these functions, go to the web pages demonstrations.wolfram.com/PlotsOfLegendrePolynomials and demonstrations.wolfram.com/ComplexSphericalHarmonics and download the demonstration in *.cdf format. The CDF Player necessary to use the demonstrations is freely available at www.wolfram.com/cdf. If you have a Mathematica licence, you can also download the source code of the respective demonstration in *.nb format.

a) Use the Legendre polynomial plotter to convince yourself that

1. the polynomials \( P_l \) with even [odd] \( l \) are symmetric [antisymmetric] with respect to the origin.
2. \( P_l(1) = 1 \) and \( P_l(x) < 1 \) for \( |x| < 1 \).
3. the polynomial \( P_l \) has exactly \( l \) roots and that there is precisely one root of the polynomial \( P_{l+1} \) between two roots of the polynomial \( P_l \)

b) The spherical harmonics demonstration plots the surface parameterized by the absolute value of the respective \( Y_{lm}(\Omega) \) and the solid angle \( \Omega \) itself by

\[
S = \{ \vec{r} \in \mathbb{R}^3 : \vec{r} = |Y_{lm}(\Omega)| \vec{e}_r \} \quad \vec{e}_r = \begin{pmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{pmatrix}
\]

The complex phase of \( Y_{lm}(\Omega) \) is coded by a rainbow color scheme. The real and imaginary part of \( Y_{lm}(\Omega) \) are shown in the same manner.

1. Convince yourself that the absolute value of \( Y_{lm} \) is always axially symmetric around \( z \).
2. Convince yourself that parity relation for inversion at the origin \( Y_{lm}(-\vec{e}_r) = (-)^l Y_{lm}(\vec{e}_r) \) holds.
3. How is \( Y_{lm} \) related to \( Y_{l,-m} \)?

Problem 3.4  Angular momentum conservation law

The angular momentum density of the electromagnetic field is defined by the antisymmetric tensor field

\[
L_{ij}(\vec{x}, t) = \frac{1}{c^2} (x_i S_j - x_j S_i),
\]

where \( \vec{S} \) denotes the Poynting vector.

a) Employ the momentum balance law to construct a local balance law for the angular momentum density of the form

\[
\partial_t L_{ij} + \nabla_k M_{ijk} = -D_{ij}.
\]

Determine the angular moment current tensor \( M_{ijk} \) as well as the mechanical torque tensor \( D_{ij} \).

Rewrite the balance law in terms of the pseudo-vector field

\[
L_i(\vec{x}, t) = \frac{1}{2} \varepsilon_{ijk} L_{jk},
\]

and suitable \( M_{ik} \) and \( D_i \).

b) Formulate the angular momentum conservation law in integral form, for \( \mathcal{L}_i = \int_V L_i \, dV \).
c) Demonstrate that in the radiation gauge, i.e., $\varphi_s = 0$, the angular momentum of the field can be decomposed, $L = L_S + L_B$, in a 'spin' part

$$L_S = \frac{1}{4\pi c^2} \int_V \vec{A} \times \dot{\vec{A}} \, dV,$$

and an 'orbital' part $L_B$ that depends explicitly on the point of reference of the coordinate system.

Due date: Tuesday, 6.11.12
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Solutions to Home assignment 3

Solution of Problem 3.1  Dirac delta function

a) Consider the charge $\Delta q$ on a spherical shell. The charge density at $(r, \theta, \varphi)$ for the point charge $\Delta q$ positioned at $(R, \theta', \varphi')$ is

$$\Delta \rho_e = \frac{\Delta q}{R^2 \sin \theta} \delta(r - R) \delta(\theta - \theta') \delta(\varphi - \varphi')$$

The charge distribution over the spherical shell is uniform. Therefore,

$$\frac{\Delta q}{R^2 \sin \theta' \Delta \theta' \Delta \varphi'} = \frac{Q}{4\pi R^2}$$

so that the charge density becomes

$$\rho_e(\vec{x}) = \frac{Q}{4\pi R^2 \sin \theta} \int_0^{2\pi} d\varphi' \int_0^{\pi} d\theta' \sin \theta' \delta(\theta - \theta') \delta(\varphi - \varphi') = \frac{Q}{4\pi} \frac{\delta(r - R)}{R^2 \sin \theta} \sin \theta = \frac{Q}{4\pi} \frac{\delta(r - R)}{R^2}$$

b) Consider the charge $\Delta q$ on a cylindrical surface. The charge density at $(\rho, \varphi, z)$ for the point charge $\Delta q$ positioned at $(b, \varphi', z')$ is

$$\Delta \rho_e = \frac{\Delta q}{b} \delta(\rho - b) \delta(\varphi - \varphi') \delta(z - z')$$

The charge distribution over the cylindrical surface is

$$\frac{\Delta q}{b \Delta \varphi' \Delta z'} = \frac{\lambda}{2\pi b}$$

while the charge density now obtains following form

$$\rho_e(\vec{x}) = \frac{\lambda}{2\pi} \frac{\delta(\rho - b)}{b} \int_0^{2\pi} d\varphi' \int_{-\infty}^{\infty} dz' \delta(z - z') \delta(\varphi - \varphi') = \frac{\lambda}{2\pi} \frac{\delta(\rho - b)}{b}$$
c) Consider the charge \( \Delta q \) on a thin disc in the \( z' = 0 \) plane. In the cylindrical coordinates the charge density at the \((\rho, \varphi, z)\) for the point charge \( \Delta q \) positioned at \((\rho', \varphi', 0)\) we can write as

\[
\Delta \rho_e = \frac{\Delta q}{\rho} \delta(\rho - \rho') \delta(\varphi - \varphi') \delta(z)
\]

The charge distribution in the disc is uniform

\[
\frac{\Delta q}{\rho' \Delta \rho' \Delta \varphi'} = \frac{Q}{R^2 \pi}, \rho' \leq R
\]

so that the total charge density equals

\[
\rho_e(\vec{x}) = \frac{Q}{R^2 \pi} \frac{\delta(z)}{\rho} \int_0^{2\pi} d\varphi' \int_0^R d\rho' \delta(\rho - \rho') \delta(\varphi - \varphi') = \left\{ \begin{array}{ll} \frac{Q}{R^2 \pi} \delta(z) & \text{if } \rho \leq R \\ 0 & \text{if } \rho > R \end{array} \right.
\]

where \( u(R - \rho) \) is the unit step function defined as

\[
\begin{array}{ll}
u(R - \rho) = 1 & \text{if } \rho \leq R \\
0 & \text{if } \rho > R
\end{array}
\]

d) Consider the charge \( \Delta q \) on a thin disc in the \( \theta' = \pi/2 \) plain. In the spherical coordinates the charge density at the \((r, \theta, \varphi)\) for the point charge \( \Delta q \) positioned at \((r', \pi/2, \varphi')\) is

\[
\Delta \rho_e = \frac{\Delta q}{r^2} \delta(r - r') \delta(\theta - \pi/2) \delta(\varphi - \varphi')
\]

The charge distribution over the disc is again uniform, thus

\[
\frac{\Delta q}{r' \Delta r' \Delta \varphi'} = \frac{Q}{R^2 \pi}, r' \leq R
\]

By substituting this equation into the expression for \( \Delta \rho_e \) we obtain

\[
\rho_e(\vec{x}) = \frac{Q}{R^2 \pi} \frac{\delta(\theta - \pi/2)}{r^2} \int_0^{2\pi} d\varphi' \int_0^R d\varphi' \delta(r - r') \delta(\varphi - \varphi') = \left\{ \begin{array}{ll} \frac{Q}{R^2 \pi} \frac{\delta(\theta - \pi/2)}{r} & \text{if } r \leq R \\ 0 & \text{if } r > R \end{array} \right.
\]

\[
= \frac{Q}{R^2 \pi} \frac{\delta(\theta - \pi/2)u(R - \rho)}{r}
\]

**Solution of Problem 3.2**

**Properties of the Dirac delta function**

a) Let \( f(x) \) be an arbitrary function. \( h(x) \) is invertible near the zero of order one at \( x_0 \).

\[
\int dx \delta[h(x)] f(x) \bigg|_{y=h(x)}^{y=h(x)} = \int dy \delta(y) f(h^{-1}(y)) \frac{1}{|h'(h^{-1}(y))|} = f(h^{-1}(0)) \frac{1}{|h'(h^{-1}(0))|}
\]

\[
\Rightarrow \delta[h(x)] = \frac{1}{|h'(x)|} \delta(x - x_0)
\]
(i) $\delta(x) = 0$ for $x \neq 0$ and $x = 0$ for $x = 0$:

$$\Rightarrow x\delta(x) = 0 \text{ for any } x$$

(ii) $\delta(x - a) = 0$ for $x \neq a$ and $\varphi(x) = \varphi(a)$:

$$\Rightarrow \varphi(x)\delta(x - a) = \varphi(a)\delta(x - a)$$

(iii) $f(x)$ is arbitrary, $\int dy f(x - y)\delta(y - z) = f(x - z)$. Now set $f(x - y) = \delta(x - y)$:

$$\Rightarrow \int dy \delta(x - y)\delta(y - z) = \delta(x - z)$$

### Solution of Problem 3.4  Angular momentum conservation law

The angular momentum density of the electromagnetic field is defined as

\[
'\text{angular momentum density}' \equiv \vec{x} \times '\text{momentum density}' \Rightarrow \vec{L} = \vec{x} \times \frac{1}{c^2} \vec{S}.
\]

The pseudo-vector $\vec{L}$ is related to a tensor of 2nd rank by the Hodge duality,

\[
L_i = \frac{1}{c^2} \varepsilon_{ijk} x_j S_k = \frac{1}{2} \varepsilon_{ijk} \frac{1}{c^2} (x_j S_k - x_k S_j) = \frac{1}{2} \varepsilon_{ijk} L_{jk}, \quad L_{ij} = \frac{1}{c^2} (x_i S_j - x_j S_i).
\]

a) Recall momentum balance (Problem 3.4),

\[
\frac{1}{c^2} \partial_t S_i + \nabla_k T_{ik} = -F_i.
\]

Poynting vector: $S_i = \frac{c}{4\pi} \varepsilon_{ijk} E_j B_k$

Maxwell stress tensor: $T_{ik} = -\frac{1}{4\pi} \left[ E_i E_k + B_i B_k - \frac{1}{2} \delta_{ik}(E^2 + B^2) \right]$

Lorentz force density: $F_i = \rho E_i + \frac{1}{c} \varepsilon_{ikl} j_k B_l$

Then,

\[
\partial_t L_{ij} = x_i \frac{1}{c^2} \partial_t S_j - x_j \frac{1}{c^2} \partial_t S_i
\]

\[
= x_i (-F_j - \nabla_k T_{jk}) - x_j (-\vec{F} - \nabla_k T_{ik})
\]

\[
= -(x_i F_j - x_j F_i) - \nabla_k (x_i T_{jk} - x_j T_{ik}) + \delta_{ik} T_{jk} - \delta_{kj} T_{ik}
\]

\[
= -D_{ij} - \nabla_k M_{ijk} + (\underbrace{T_{ji} - T_{ij}}_{\text{M}_{ij}})
\]

\[
= 0 \text{ by symmetry}
\]

Thus, $\partial_t L_{ij} + \nabla_k M_{ijk} = -D_{ij}$, where we have introduced the

\[
\text{torque density: } D_{ij} = x_i F_j - x_j F_i,
\]

\[
\text{angular momentum current: } M_{ijk} = x_i T_{jk} - x_j T_{ik}.
\]
Note the symmetries $D_{ij} = -D_{ji}$, $M_{ijk} = -M_{jik}$. The Hodge duality yields pseudo-vectors and -tensors,

$$D_i = \frac{1}{2} \varepsilon_{ijk} D_{jk}, \quad M_{ik} = \frac{1}{2} \varepsilon_{ijn} M_{mnk},$$

then

$$\partial_i L_i + \nabla_k M_{ik} = -D_i$$

b) The angular momentum conservation law in integral form reads

$$\frac{d}{dt} \int_V L_i \, dV + \int_V D_i \, dV = -\int_V \nabla_k M_{ik} \, dV = -\int_{\partial V} M_{ik} \, df_k$$

Interpretation: The angular momentum of the field can be converted to mechanical angular momentum or flow through the surface.

c) The radiation gauge, $\varphi_s = 0$, implies

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad \text{and} \quad \vec{E} = -\frac{1}{c} \dot{\vec{A}}.$$ 

Poynting vector

$$S_k = \frac{c}{4\pi} (\vec{E} \times \dot{\vec{B}})_k = \frac{c}{4\pi} (\vec{E} \times \vec{\nabla} \times \vec{A})_k = \frac{c}{4\pi} E_l \nabla_k A_l - \frac{c}{4\pi} E_l \nabla_l A_k$$

$$L_i = \int_V L_i \, dV = \frac{1}{c^2} \int_V \varepsilon_{ijk} x_j S_k \, dV$$

$$= \frac{1}{4\pi c} \int_V \varepsilon_{ijk} x_j (E_l \nabla_k A_l - E_l \nabla_l A_k) \, dV \quad \text{partial integration}$$

$$= -\frac{1}{4\pi c} \int_V \varepsilon_{ijk} (A_l \nabla_k x_j E_l - A_k \nabla_l x_j E_l) \, dV + \text{surface terms}$$

$$= -\frac{1}{4\pi c} \int_V \varepsilon_{ijk} (A_l \delta_{kj} E_l + A_l x_j \nabla_k E_l - A_k \delta_{ij} E_l - A_k x_j \nabla_l E_l) \, dV + \text{surface terms}$$

$$= -\frac{1}{4\pi c} \int_V \left[ A_l (\vec{x} \times \vec{\nabla})_l E_l + (\vec{A} \times \vec{x}) E_l + (\vec{A} \times \vec{x}) \cdot \text{div} \vec{E} \right] \, dV$$

Which may be rewritten as $\vec{L} = \vec{L}_S + \vec{L}_B$ with a part independent of the point of reference (‘spin’),

$$\vec{L}_S = \frac{1}{4\pi c^2} \int_V \vec{A} \times \dot{\vec{A}} \, dV,$$

and an orbital part,

$$\vec{L}_B = \frac{1}{4\pi c} \int_V \left[ (\vec{x} \times \vec{A}) \text{div} \vec{E} - \vec{A} \cdot (\vec{x} \times \vec{\nabla}) \vec{E} \right] \, dV.$$