

Three useful bounds in quantum mechanics — easily obtained by Wiener integration

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Abstract

In a reasonably self-contained and explicit presentation we illustrate the efficiency of the Feynman–Kac formula for the rigorous derivation of three inequalities of interest in non-relativistic quantum mechanics.

Keywords: Wiener integration, Feynman–Kac formula, diamagnetic monotonicities, integrated density of states

“As it was, as soon as I heard Feynman describe his path integral approach to quantum mechanics during a lecture at Cornell everything became clear at once; and there remained only matters of rigor.” [1]

“Hardly a month passes without someone discovering yet another application [of the Feynman–Kac formula]. Remarkable and, of course, highly gratifying.” [2]

Mark Kac (1914–1984)

1 Feynman–Kac formula

In order to model the phenomenon of Brownian motion Norbert Wiener [3] introduced as early as 1923 a certain probability distribution μ concentrated on the set W^d of continuous paths $w : [0, \infty[\rightarrow \mathbb{R}^d, t \mapsto w(t)$ from the positive

half-line $[0, \infty[\subset \mathbb{R}^1$ into d -dimensional Euclidean space \mathbb{R}^d , $d \in \{1, 2, 3, \dots\}$, which start at the origin, $w(0) = 0$. This distribution is a mathematically well-defined positive measure in the sense of general measure theory [4] and therefore induces a corresponding concept of integration over paths, which we denote by $\int_{W^d} \mu(dw)(\cdot)$. The (standard) *Wiener measure* μ is uniquely determined by requiring that it is *Gaussian* with normalization

$$\boxed{\int_{W^d} \mu(dw) = 1} \quad (1)$$

and first, respectively, second moments given by

$$\boxed{\int_{W^d} \mu(dw) w_j(t) = 0, \quad \int_{W^d} \mu(dw) w_j(t) w_k(s) = \delta_{jk} \min\{t, s\}} \quad (2)$$

for all $j, k \in \{1, \dots, d\}$ and all $t, s \in [0, \infty[$. Here $w_j(t)$ denotes the j -th component of the path w evaluated at (time) parameter $t \geq 0$. The simple Wiener integrals (1) and (2) imply that the components of w are, in probabilistic language, centered, independent and identically distributed.

To Wiener integrals apply, in contrast to Feynman path integrals, all the rules and computational tools provided by general measure and integration theory, most notably Lebesgue's dominated-convergence theorem [4]. As a consequence, Wiener integration often serves, via the Feynman–Kac formula, as an efficient technique for obtaining results in quantum mechanics with complete rigour. An impressive compilation of such results was given by Barry Simon already in 1979. Since then not much has changed which is reflected by the fact that the second edition of his book [5] differs from the first one only by an addition of bibliographic notes on some of the more recent developments. Still Wiener integration should be considered neither as a secret weapon nor as a panacea for obtaining rigorous results in quantum mechanics. In any case, the Feynman–Kac formula is more than just a poetic rewriting of a Lie–Trotter formula. Ironically, Richard Feynman himself took advantage of that as early as 1955 in his celebrated paper on the polaron [6], in particular, by using the Jensen inequality.

Now, what is the Feynman–Kac formula? Let us consider a spinless charged particle with configuration space \mathbb{R}^d subjected to a scalar potential $v : \mathbb{R}^d \rightarrow \mathbb{R}^1, q \mapsto v(q), q = (q_1, \dots, q_d)$ and a vector potential $a : \mathbb{R}^d \rightarrow \mathbb{R}^d, q \mapsto a(q), a = (a_1, \dots, a_d)$. The latter generates a magnetic field (tensor) defined by $b_{jk} := \partial a_k / \partial q_j - \partial a_j / \partial q_k$. The corresponding (non-relativistic) quantum system is informally given by the Hamiltonian

$$\boxed{H(a, v) := (P - a(Q))^2 / 2 + v(Q)} \quad (3)$$

where $Q = (Q_1, \dots, Q_d)$ and $P = (P_1, \dots, P_d)$ denote the d -component operators of position and canonical momentum, respectively. They obey the canonical commutation relations $Q_j P_k - P_k Q_j = i\hbar \delta_{jk} \mathbb{1}$. Here $i = \sqrt{-1}$ is the imaginary unit and $\hbar > 0$ is Planck's constant (divided by 2π). Moreover, we have chosen physical units where both the mass and the charge of the particle are equal to 1. Under rather weak assumptions [7] on a and v , $H(a, v)$ can be defined as a self-adjoint operator on the Hilbert space $L^2(\mathbb{R}^d)$ of all (equivalence classes of) Lebesgue square-integrable complex-valued functions on \mathbb{R}^d . Furthermore its ‘‘Boltzmann–Gibbs operator’’ $e^{-\beta H(a, v)}$ even possesses for each $\beta \in]0, \infty[$ an integral kernel $\langle q | e^{-\beta H(a, v)} | q' \rangle$ (in other words, position representation or Euclidian propagator) which is jointly continuous in $q, q' \in \mathbb{R}^d$.

We are now prepared to state the *Feynman–Kac formula*. Apart from mathematical subtleties its content is most concisely expressed by the following representation free version:

$$e^{-\beta H(a, v)} = \int_{W^d} \mu(dw) e^{-i w(\beta \hbar^2) \cdot P / \hbar} \exp \left\{ - \int_0^\beta d\tau v(w(\tau \hbar^2) \mathbb{1} + Q) \right\} \\ \times \exp \left\{ i \hbar \int_0^\beta d\tau \dot{w}(\tau \hbar^2) \cdot a(w(\tau \hbar^2) \mathbb{1} + Q) \right\}. \quad (4)$$

Here the dot ‘‘ \cdot ’’ between two d -component quantities refers to the Euclidian scalar product of \mathbb{R}^d and the integral containing the vector potential is a suggestive notation for a *stochastic line integral* in the sense of R. L. Stratonovich and D. L. Fisk (corresponding to a mid-point discretization). In (4) the Wiener integration serves to disentangle the non-commuting operators P and Q in $e^{-\beta H(a, v)}$. For related remarks see Ref. [8] and references therein. We recall from (2) that μ neither depends on β nor on \hbar . Also P/\hbar is independent of \hbar .

By going informally to the position representation of (4) one gets the rigorously proven formula

$$\langle q | e^{-\beta H(a, v)} | q' \rangle = \int_{W^d} \mu(dw) \delta(w(\beta \hbar^2) + q' - q) \exp \left\{ - \int_0^\beta d\tau v(w(\tau \hbar^2) + q') \right\} \\ \times \exp \left\{ i \hbar \int_0^\beta d\tau \dot{w}(\tau \hbar^2) \cdot a(w(\tau \hbar^2) + q') \right\}. \quad (5)$$

It even holds for a class of potentials v for which $H(a, v)$ is not bounded from below [7] and remains true (by the self-adjointness of $H(a, v)$), if on its right side q and q' are exchanged and simultaneously i is changed to $-i$. The

Dirac delta in (5) indicates that all paths w to be integrated over arrive in $q - q' \in \mathbb{R}^d$ at “time” $\beta\hbar^2 > 0$. In fact, they may be considered to end there, because μ is Markovian and the Wiener integrand in (5) does not depend on $w(\tau\hbar^2)$ for $\tau > \beta$. More precisely, the path integration may be performed with respect to the *Brownian bridge* [5, 7]. In the Appendix below we shall present what we think is an illuminating derivation of the “bridge version” of (5), although in the following we shall not make (explicit) use of that version.

In the next three sections we are going to illustrate the usefulness of (5) by deriving three inequalities of interest in quantum mechanics.

2 Diamagnetic inequality

Theorem 2.1.

$$\left| \langle q | e^{-\beta H(a,v)} | q' \rangle \right| \leq \langle q | e^{-\beta H(0,v)} | q' \rangle \quad (6)$$

holds for all $\beta > 0$ and all $q, q' \in \mathbb{R}^d$.

Proof. Inequality (6) is an immediate consequence of (5) by taking the absolute value, applying the “triangle inequality”

$$\left| \int_{W^d} \mu(dw) (\cdot) \right| \leq \int_{W^d} \mu(dw) |(\cdot)| \quad (7)$$

and using the elementary identity $|e^{x+iy}| = e^x$ for $x, y \in \mathbb{R}^1$. □

Remarks:

- (i) This elegant proof is due to Edward Nelson, see Ref. [9] (also for other historical aspects of (6) and related inequalities).
- (ii) If $\int_{\mathbb{R}^d} dq e^{-\beta v(q)} < \infty$, the *free energy* $-\beta^{-1} \ln \int_{\mathbb{R}^d} dq \langle q | e^{-\beta H(0,v)} | q \rangle$ at inverse temperature $\beta > 0$ exists and (6) then implies that it cannot be lowered by turning on a magnetic field. Under weaker assumptions on v , for example for the hydrogen atom (that is, for $d = 3$ and $v(q) = -\gamma/|q|$ with $\gamma > 0$) (6) still implies in the limit $\beta \rightarrow \infty$ the same sort of stability for the *ground-state energy*. Altogether this explains the name *diamagnetic inequality*.
- (iii) There are also diamagnetic inequalities in case the particle is restricted to a region in \mathbb{R}^d of finite volume with Dirichlet, Neumann or other boundary conditions [10, 11]. Moreover, the proof of the diamagnetic

inequality easily extends to the case of many (interacting) particles, provided there is no spin and no Fermi statistics involved.

An interesting question is what can be said if $a \neq 0$ is changed (pointwise) to another vector potential $a' \neq 0$. For a partial answer see Sec. 4 below.

3 Quasi-classical upper bound on the integrated density of states in the case of a random scalar potential

In the single-particle theory of electronic properties of disordered or amorphous solids the scalar potential v in $H(a, v)$ is considered to be a realization of a *random* field on \mathbb{R}^d which is distributed according to some probability measure ν on some set V of potentials v . We denote by $\int_V \nu(dv) (\cdot)$ the corresponding (functional) integration or averaging. One example is a *Gaussian* ν with vanishing first moments and second moments given by $\int_V \nu(dv) v(q)v(q') = C(q - q')$ for all $q, q' \in \mathbb{R}^d$ with some (even) covariance function $C : \mathbb{R}^d \rightarrow \mathbb{R}^1$. The fact that the second moments only depend on the difference $q - q'$ reflects the assumed “homogeneity on average”. We also assume that C is continuous, $C(q)$ tends to zero as $|q| \rightarrow \infty$ and the single-site variance obeys $0 < C(0) < \infty$. The \mathbb{R}^d -homogeneity together with the decay of the correlations of the fluctuations at different sites with increasing distance implies the \mathbb{R}^d -*ergodicity* of the (Gaussian) random potential.

A quantity of basic interest in the above-mentioned theory is the *integrated density of states*. It may be defined [12, 13, 7] as the non-decreasing function $N : \mathbb{R}^1 \rightarrow \mathbb{R}^1, E \mapsto N(E, a, q)$ where

$$N(E, a, q) := \int_V \nu(dv) \langle q | \Theta(E - H(a, v)) | q \rangle. \quad (8)$$

Here Θ denotes Heaviside’s unit-step function and the (non-random) vector potential a as well as the position $q \in \mathbb{R}^d$ are considered as parameters. If the random potential (characterized by ν) and the magnetic field (generated by a) are both homogeneous, then $N(E, a, q)$ actually does not depend on q . Of course, in the physically most relevant cases the random potential should be even ergodic, so that $N(E, a, 0)$ coincides for ν -almost all realizations v with the number of eigenvalues per volume of a finite-volume restriction of $H(a, v)$ below the energy $E \in \mathbb{R}^1$ in the infinite-volume limit. Nevertheless, the following estimate holds also for random potentials and magnetic fields which are not homogeneous.

Theorem 3.1. *If the probability measure ν of the random potential has the property¹ that $L_\beta := \text{ess sup}_{r \in \mathbb{R}^d} \int_V \nu(\mathrm{d}v) e^{-\beta v(r)} < \infty$ for all $\beta > 0$, then*

$$N(E, a, q) \leq (2\pi\beta\hbar^2)^{-d/2} L_\beta e^{\beta E} \quad (9)$$

holds for all energies $E \in \mathbb{R}^1$ and all $\beta > 0$.

Proof.

$$N(E, a, q) e^{-\beta E} \leq \int_V \nu(\mathrm{d}v) \langle q | e^{-\beta H(a, v)} | q \rangle \leq \int_V \nu(\mathrm{d}v) \langle q | e^{-\beta H(0, v)} | q \rangle \quad (10)$$

$$= \int_{W^d} \mu(\mathrm{d}w) \delta(w(\beta\hbar^2)) \int_V \nu(\mathrm{d}v) \exp \left\{ - \int_0^\beta \mathrm{d}\tau v(w(\tau\hbar^2) + q) \right\} \quad (11)$$

$$\leq \int_0^\beta \frac{\mathrm{d}\tau}{\beta} \int_{W^d} \mu(\mathrm{d}w) \delta(w(\beta\hbar^2)) \int_V \nu(\mathrm{d}v) e^{-\beta v(w(\tau\hbar^2) + q)} \quad (12)$$

$$\leq (2\pi\beta\hbar^2)^{-d/2} L_\beta. \quad (13)$$

Here (10) is due to the elementary inequality $\Theta(E - H(a, v)) \leq e^{\beta(E - H(a, v))}$, referring to the spectral theorem, and (6). Eq. (5) then gives (11). The next inequality is Jensen's with respect to the uniform average $\beta^{-1} \int_0^\beta \mathrm{d}\tau (\cdot)$. The claim now follows from the definition of L_β , Eq. (5) with $(a, v) = (0, 0)$ and $\int_0^\beta \mathrm{d}\tau = \beta$. The various interchanges of integrations can be justified by the Fubini-Tonelli theorem [4]. \square

Remarks:

- (i) Theorem 3.1 is a slight extension of a result which goes back to Pastur, see Thm. 9.1 in Ref. [12]. The right side of (9) is *quasi-classical* in the sense that it does not depend on a and does not take into account, due to the Jensen inequality (13) in its proof, the non-commutativity of the kinetic and potential energy.
- (ii) While the estimate (9) holds for rather general random potentials, the various inequalities in its proof are responsible for its roughness, even when optimized with respect to $\beta > 0$. Nevertheless, it shows that $N(E)$ decreases to 0 at least exponentially fast as $E \rightarrow -\infty$. For a homogeneous Gaussian random potential (and a constant magnetic field) the optimized estimate even reflects the exact Gaussian decay

¹Given a function $f : \mathbb{R}^d \rightarrow \mathbb{R}^1$, then $\text{ess sup}_{r \in \mathbb{R}^d} |f(r)|$ denotes the smallest $M \in [0, \infty]$ such that $|f(r)| \leq M$ holds for Lebesgue-almost all $r \in \mathbb{R}^d$.

[12, 13] $\ln N(E) \sim -E^2/2C(0)$ as $E \rightarrow -\infty$. For a non-Gaussian random potential, like a repulsive Poissonian one, the leading low-energy decay of N , the *Lifshitz tail*, typically is of true quantum nature [12, 13, 14] and can therefore not be reflected by the right side of (9). Although the (universal) leading high-energy growth $N(E) \sim (E/2\pi\hbar^2)^{d/2}/\Gamma(1+d/2)$ as $E \rightarrow \infty$, see Refs. [12, 13] and references therein, is quasi-classical, the optimized right side of (9) overestimates it slightly by a constant factor (due to the elementary inequality in (10)).

- (iii) For related quasi- and pseudo-classical bounds (with a non-random v) we refer to Sec. 9 in Simon's book [5] and to Ref. [15].

4 A simple diamagnetic monotonicity

For a partial answer to the question raised at the end of Sec. 2 we only consider the planar case $d = 2$ with $v = 0$ and a perpendicular magnetic field not depending on the second co-ordinate q_2 . We assume that $b := b_{12}$ is a continuously differentiable function of $q_1 \in \mathbb{R}^1$. One possible vector potential generating b , not depending on q_2 either, is given by $a^{(b)}(q) := (0, \int_0^{q_1} dr b(r))$. In the following theorem $H^{(b)}$ denotes any Hamiltonian on $L^2(\mathbb{R}^2)$ which is gauge equivalent to $H(a^{(b)}, 0) = P_1^2/2 + (P_2 - a_2^{(b)}(Q_1))^2/2$ for the given b . The assertion (14) is therefore gauge invariant. Nevertheless, in the proof we will use $H(a^{(b)}, 0)$ and see that one can dispense with the absolute value on the right side of (14) in this particular gauge.

Theorem 4.1. *If b and B are two magnetic fields as just described and satisfy either $|b(r)| \leq B(r)$ or $|b(r)| \leq -B(r)$ for all $r \in \mathbb{R}^1$, then*

$$\left| \langle q | e^{-\beta H^{(B)}} | q' \rangle \right| \leq \left| \langle (q_1, 0) | e^{-\beta H^{(b)}} | (q'_1, 0) \rangle \right| e^{-(q_2 - q'_2)^2 / (2\beta\hbar^2)} \quad (14)$$

holds for all $\beta > 0$, all $q = (q_1, q_2) \in \mathbb{R}^2$ and all $q' = (q'_1, q'_2) \in \mathbb{R}^2$.

Proof. By (5) the left side of (14) is invariant under a global sign change of B . Therefore it suffices to consider the case $B(r) \geq 0$. For notational transparency we put $\hbar = 1$ and write a and A for $a_2^{(b)}$ and $a_2^{(B)}$, respectively. For a given pair $(\beta, w) \in]0, \infty[\times W^1$ we introduce the notations

$$m_\beta(a, w) := \beta^{-1} \int_0^\beta d\tau a(w(\tau)), \quad (15)$$

$$s_\beta^2(a, w) := \beta^{-1} \int_0^\beta d\tau (a(w(\tau)))^2 - (m_\beta(a, w))^2 \quad (16)$$

for the mean and variance of $a(w(\tau))$ with respect to the uniform average $\beta^{-1} \int_0^\beta d\tau (\cdot)$, and similarly with A instead of a . Next we observe the following two “doubling identities”

$$2\beta^2 [s_\beta^2(A, w) - s_\beta^2(a, w)] \quad (17)$$

$$= \int_0^\beta d\tau \int_0^\beta d\sigma \left\{ [A(w(\tau)) - A(w(\sigma))]^2 - [a(w(\tau)) - a(w(\sigma))]^2 \right\} \quad (18)$$

$$= \int_0^\beta d\tau \int_0^\beta d\sigma [a_+(w(\tau)) - a_+(w(\sigma))] [a_-(w(\tau)) - a_-(w(\sigma))]. \quad (19)$$

The last integrand is non-negative, because the two functions $r \mapsto a_\pm(r) := A(r) \pm a(r) = \int_0^r dr' (B(r') \pm b(r'))$, $r \in \mathbb{R}^1$, are both non-decreasing since $B(r') \geq |b(r')| \geq \mp b(r')$ by assumption. The same arguments apply when the path w is replaced by the rigidly shifted one $w + q'_1 \in W^1 + \mathbb{R}^1$ defined by $(w + q'_1)(\tau) := w(\tau) + q'_1$. To summarize, we have shown so far that

$$s_\beta^2(a, w + q'_1) \leq s_\beta^2(A, w + q'_1). \quad (20)$$

Since $H^{(B)}$, in the particular gauge chosen, commutes with P_2 , we have

$$H^{(B)} = \int_{\mathbb{R}^1} dk H^{(B)}(k) \otimes |k\rangle\langle k|, \quad (21)$$

using an informal notation for a direct-integral decomposition. Here the one-parameter family of effective Hamiltonians

$$H^{(B)}(k) := P_1^2/2 + (k\mathbb{1} - A(Q_1))^2/2, \quad k \in \mathbb{R}^1, \quad (22)$$

acts on the Hilbert space $L^2(\mathbb{R}^1)$ of (wave) functions of the first co-ordinate. By (21) and (22) we get

$$\langle q|e^{-\beta H^{(B)}}|q'\rangle = (2\pi)^{-1} \int_{\mathbb{R}^1} dk \langle q_1|e^{-\beta H^{(B)}(k)}|q'_1\rangle e^{ik(q_2 - q'_2)} \quad (23)$$

$$= (2\pi\beta)^{-1/2} e^{-(q_2 - q'_2)^2/(2\beta)} \int_{W^1} \mu(dw) \delta(w(\beta) + q'_1 - q_1) \times \exp \left\{ -\beta s_\beta^2(A, w + q'_1)/2 \right\} \exp \left\{ i(q_2 - q'_2)m_\beta(A, w + q'_1) \right\}. \quad (24)$$

Here we have used (5) with $d = 1$, $a = 0$ and $v = (k - A)^2/2$ and then performed the (Gaussian) integration with respect to k . By applying the “triangle inequality” to (24) and then using (20) we finally obtain

$$\begin{aligned}
& e^{(q_2 - q'_2)^2 / (2\beta)} \left| \langle q | e^{-\beta H^{(B)}} | q' \rangle \right| \\
& \leq (2\pi\beta)^{-1/2} \int_{W^1} \mu(dw) \delta(w(\beta) + q'_1 - q_1) \exp \left\{ -\beta s_\beta^2(a, w + q'_1) / 2 \right\} \quad (25)
\end{aligned}$$

$$= \langle (q_1, 0) | e^{-\beta H^{(b)}} | (q'_1, 0) \rangle = \left| \langle (q_1, 0) | e^{-\beta H^{(b)}} | (q'_1, 0) \rangle \right|. \quad (26)$$

The last two equalities follow again from (24) with a instead of A . \square

Remarks:

- (i) To our knowledge, Theorem 4.1 first appeared in Ref. [16]. It complements some of the results obtained by Loss, Thaller and Erdős [17, 18]. For a survey of results of this genre see Sec. 9 in Ref. [19].
- (ii) For a given sign-definite B the right side of (14) can be made explicit by choosing for b the globally constant field $B_0 := \inf_{r \in \mathbb{R}^1} |B(r)|$, so that

$$\left| \langle q | e^{-\beta H^{(B)}} | q' \rangle \right| \leq \frac{B_0 / 4\pi\hbar}{\sinh(\beta\hbar B_0 / 2)} \exp \left\{ -\frac{(q_1 - q'_1)^2 B_0 / 4\hbar}{\tanh(\beta\hbar B_0 / 2)} - \frac{(q_2 - q'_2)^2}{2\beta\hbar^2} \right\}.$$

If $B_0 \neq 0$, the Gaussian decay on the right side is faster along the 1- than along the 2-direction. Such an anisotropy has been found also for the almost-sure transport properties in the case that B is a (Gaussian) random field with non-zero mean [20].

Appendix

For convenience of the reader we are going to derive the bridge version of the Feynman–Kac formula (5). We start out from a fixed triple $(T, q, q') \in]0, \infty[\times \mathbb{R}^d \times \mathbb{R}^d$ and associate to each continuous path $w : [0, \infty[\rightarrow \mathbb{R}^d$ with $w(0) = 0$ another path $\hat{w} : [0, T] \rightarrow \mathbb{R}^d$ defined by

$$\hat{w}(t) := w(t) + q' - \frac{t}{T}(w(T) + q' - q), \quad t \in [0, T]. \quad (27)$$

Obviously, \hat{w} is a *bridge path* in the sense that $\hat{w} \in \Omega_{T, q, q'}^d$ where

$$\Omega_{T, q, q'}^d := \left\{ \omega : [0, T] \rightarrow \mathbb{R}^d \mid \omega \text{ is continuous, } \omega(0) = q', \omega(T) = q \right\} \quad (28)$$

is the set of all continuous paths connecting position q' to position q in the time period T . In fact, $\Omega_{T, q, q'}^d$ is the image of W^d under the mapping $w \mapsto \hat{w}$ given by (27).

Writing $\langle(\cdot)\rangle := \int_{W^d} \mu(dw)(\cdot)$ for the expectation or averaging induced by the Wiener measure, Eqs. (1) and (2) yield

$$\langle \widehat{w}_j(t) \rangle = q'_j - \frac{t}{T}(q'_j - q_j), \quad (29)$$

$$\langle \widehat{w}_j(t) \widehat{w}_k(s) \rangle - \langle \widehat{w}_j(t) \rangle \langle \widehat{w}_k(s) \rangle = \delta_{jk} \left(\min\{t, s\} - \frac{ts}{T} \right) \quad (30)$$

for the first and second moments of the bridge paths (27), in other words, for their mean and covariance.

Hence, by the (affine) linearity and the surjectivity of the mapping $w \mapsto \widehat{w}$ the induced image of the Wiener measure μ on W^d is a *Gaussian* probability measure $\rho_{T,q,q'}$ on $\Omega_{T,q,q'}^d$ with mean and covariance given by the right sides of (29) and (30). Other useful consequences of (2) and (27) are the two equalities

$$\langle w_k(T) \widehat{w}_j(t) \rangle = 0 = \langle w_k(T) \rangle \langle \widehat{w}_j(t) \rangle. \quad (31)$$

They hold for all $j, k \in \{1, \dots, d\}$ and all $t \in [0, T]$ and imply, by the Gaussian nature of the Wiener measure μ , that the random point $w(T) \in \mathbb{R}^d$ is independent of the family of random points $\{\widehat{w}(t)\}_{t \in [0, T]} \subset \mathbb{R}^d$.

Now, how can all this be applied to the right side of (5)? First we get from (27) that

$$\delta(w(T) + q' - q) F_T(w + q') = \delta(w(T) + q' - q) F_T(\widehat{w}) \quad (32)$$

for any complex-valued function(al) F_T on $W^d + \mathbb{R}^d$, not depending on the points $w(t)$ of w for $t > T$. The two mentioned independencies then give

$$\langle \delta(w(T) + q' - q) F_T(w + q') \rangle = \langle \delta(w(T) + q' - q) \rangle \langle F_T(\widehat{w}) \rangle \quad (33)$$

$$= (2\pi T)^{-d/2} e^{-(q-q')^2/2T} \langle F_T(\widehat{w}) \rangle \quad (34)$$

and the *bridge version* of the Feynman–Kac formula (5) eventually reads

$$\begin{aligned} \langle q | e^{-\beta H(a,v)} | q' \rangle &= (2\pi\beta\hbar^2)^{-d/2} e^{-(q-q')^2/2\beta\hbar^2} \\ &\times \int_{\Omega_{\beta\hbar^2, q, q'}^d} \rho_{\beta\hbar^2, q, q'}(d\omega) \exp \left\{ - \int_0^\beta d\tau v(\omega(\tau\hbar^2)) \right\} \\ &\times \exp \left\{ i\hbar \int_0^\beta d\tau \dot{\omega}(\tau\hbar^2) \cdot a(\omega(\tau\hbar^2)) \right\}. \quad (35) \end{aligned}$$

Remaining matters of rigour can be supplied.

We note that a “one-parameter decomposition” similar to (27) for a Gaussian random potential with non-negative covariance function has turned out to be useful in proving local Lipschitz continuity of the corresponding integrated density of states (8), see the proof of Cor. 4.3 in Ref. [10].

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