A rigorous path integral for quantum spin using flat-space Wiener regularization

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Adapting ideas of Daubechies and Klauder [J. Math. Phys. **26**, 2239 (1985)] we derive a rigorous continuum path-integral formula for the semigroup generated by a spin Hamiltonian. More precisely, we use spin coherent vectors parametrized by complex numbers to relate the coherent representation of this semigroup to a suitable Schrödinger semigroup on the Hilbert space $L^2(\mathbb{R}^2)$ of Lebesgue square-integrable functions on the Euclidean plane $\mathbb{R}^2$. The path-integral formula emerges from the standard Feynman–Kac–Itô formula for the Schrödinger semigroup in the ultradiffusive limit of the underlying Brownian bridge on $\mathbb{R}^2$. In a similar vein, a path-integral formula can be constructed for the coherent representation of the unitary time evolution generated by the spin Hamiltonian. © *1999 American Institute of Physics.* [S0022-2488(99)02005-8]

I. INTRODUCTION

Even 50 years after the appearance of Feynman's celebrated paper[1] that introduced the path-integral formalism[2–6] into quantum theory in a heuristic but convincing manner, there is no general consensus on how to treat a quantum spin within this framework. To the best of our knowledge, among the various approaches over the years, see, for example, Refs. 7–22, the only rigorous expression for the dynamics of a quantum spin in terms of an integral over continuous paths is due to Daubechies and Klauder.[12] These authors were able to write the coherent representation of the unitary time-evolution operator of a spin with a definite quantum number as a Wiener-regularized path integral, more precisely, as the ultradiffusive limit of a well-defined integral over spherical Brownian-motion paths.

The main goal of the present paper is to show that one may equally well perform the Wiener regularization by employing planar Brownian motion. In this way also a closer contact to symbolic continuum path-integral formulas widely discussed in the recent literature[23–28] is established. One may hope that the wealth of analytical tools associated with the flat-space Wiener measure helps clarifying some subtle points there.

II. BASIC DEFINITIONS, RESULT, AND COMMENTS

We consider a single spin with fixed *quantum number* $j \in \{0,1/2,1,3/2,...\}$, that is, using physical units where Planck's constant $2\pi\hbar$ equals 2π,

$$\tfrac{1}{2}(\mathcal{J}_+\mathcal{J}_- + \mathcal{J}_-\mathcal{J}_+) + \mathcal{J}_3^2 = j(j+1)\mathbf{1}. \tag{1}$$

The *spin operators* $\mathcal{J}_+$, $\mathcal{J}_-$, and $\mathcal{J}_3$ obey the usual angular-momentum commutation relations $\mathcal{J}_+\mathcal{J}_- - \mathcal{J}_-\mathcal{J}_+ = 2\mathcal{J}_3$, $\mathcal{J}_3\mathcal{J}_\pm - \mathcal{J}_\pm\mathcal{J}_3 = \pm\mathcal{J}_\pm$ and are viewed as acting on the $(2j+1)$-

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TABLE I. Contravariant symbols for selected operators on $\mathbb{C}^{2j+1}$, which are bounded and continuous.

Operator	Contravariant symbol	Operator	Contravariant symbol
$\mathcal{J}_+$	$2(j+1)\dfrac{z^*}{1+\lvert z\rvert^2}$	$\mathcal{J}_+\mathcal{J}_-$	$-2(j+1)\dfrac{1-2(j+1)\lvert z\rvert^2}{(1+\lvert z\rvert^2)^2}$
$\mathcal{J}_-$	$2(j+1)\dfrac{z}{1+\lvert z\rvert^2}$	$\mathcal{J}_-\mathcal{J}_+$	$2(j+1)\dfrac{2(j+1)\lvert z\rvert^2-\lvert z\rvert^4}{(1+\lvert z\rvert^2)^2}$
$\mathcal{J}_3$	$-(j+1)\dfrac{1-\lvert z\rvert^2}{1+\lvert z\rvert^2}$	$\mathcal{J}_3^2$	$(j+1)(j+\tfrac{3}{2})\left(\dfrac{1-\lvert z\rvert^2}{1+\lvert z\rvert^2}\right)^2-\dfrac{j+1}{2}$

dimensional complex Hilbert space $\mathbb{C}^{2j+1}$. Its standard scalar product is denoted as $\langle\cdot|\cdot\rangle$ and, by convention, antilinear in the first argument. The unit operator on $\mathbb{C}^{2j+1}$ is denoted by **1**.

Non-normalized so-called *coherent vectors*[9,29] in this Hilbert space,

$$|z\rangle := g(z)e^{z\mathcal{J}_+}|j,-j\rangle, \quad z\in\mathbb{C}, \tag{2}$$

are parametrized by complex numbers z. Henceforth, z^* will refer to their complex conjugates, $z_1 := (z+z^*)/2$ and $z_2 := (z-z^*)/2i$ to their real and imaginary parts, and we write $f^*(z) := (f(z))^*$ for the values of complex-conjugated functions f^*. For later notational convenience the strictly positive prefactor is taken as

$$g(z) := \left(\frac{2j+1}{\pi}\right)^{1/2}(1+|z|^2)^{-j-1}, \tag{3}$$

and a normalized *spin-down vector* $|j,-j\rangle\in\mathbb{C}^{2j+1}$, obeying $\mathcal{J}_-|j,-j\rangle=0$ and $\langle j,-j|j,-j\rangle=1$, serves as the reference vector. Every vector $|\psi\rangle\in\mathbb{C}^{2j+1}$ is characterized by its so-called *coherent representation* $\langle z|\psi\rangle$, a function of the form $g(z)$ times a polynomial in z^* of maximal degree $2j$. The scalar product of two coherent vectors $\langle z|z'\rangle = g(z)g(z')(1+z^*z')^{2j}$ is an example. Given an arbitrary operator $\mathcal{B}$ on $\mathbb{C}^{2j+1}$, the scalar product $\langle z|\mathcal{B}|z'\rangle$ of $|z\rangle$ and $\mathcal{B}|z'\rangle$ is called the *coherent representation* of $\mathcal{B}$. The mapping $(z,z')\mapsto\langle z|\mathcal{B}|z'\rangle$ is continuous, because $z\mapsto|z\rangle$ is continuous, every operator $\mathcal{B}$ on $\mathbb{C}^{2j+1}$ is bounded, and the scalar product $(|\varphi\rangle,|\psi\rangle)\mapsto\langle\varphi|\psi\rangle$ is continuous. An example is $\langle z|e^{2\lambda\mathcal{J}_3}|z'\rangle = g(z)g(z')(e^{-\lambda}+z^*z'e^{\lambda})^{2j}$, $\lambda\in\mathbb{C}$.

In what follows, it is a comforting fact that whatever the *spin Hamiltonian* $\mathcal{H}$ may be—given as a (self-adjoint) operator on $\mathbb{C}^{2j+1}$—it is polynomial in the spin operators $\mathcal{J}_+$, $\mathcal{J}_-$, and $\mathcal{J}_3$, and it is always possible to write it in *pseudodiagonal form*,

$$\mathcal{H} = \int_{\mathbb{C}} d^2z\, h(z)|z\rangle\langle z|. \tag{4}$$

Here the (real-valued) function h on $\mathbb{C}\cong\mathbb{R}\times\mathbb{R}$ may be chosen bounded and continuous,[30–32] the operator $|z\rangle\langle z|/\langle z|z\rangle$ denotes the orthogonal projection onto the one-dimensional subspace spanned by $|z\rangle\in\mathbb{C}^{2j+1}$, and $d^2z := dz_1\,dz_2$ is the two-dimensional Lebesgue measure on the Euclidean plane $\mathbb{R}\times\mathbb{R} =: \mathbb{R}^2$. Following Ref. 33, we call h a *contravariant symbol* of $\mathcal{H}$, elsewhere called an upper[34] or lower[35] symbol. In particular, the unit operator **1** has the constant 1 as a contravariant symbol. In this sense, the coherent vectors are *unity-resolving* and hence (over-)complete. Other examples for contravariant symbols are listed in Table I; confer Ref. 36.

After these preparations we are able to state the main result of the present paper, namely, a rigorous expression for the *spin semigroup* $\{e^{-t\mathcal{H}}\}_{t\geqslant 0}$ as the *ultradiffusive limit* of a Wiener type of integral over Brownian-motion paths $\{s \mapsto b(s) = b_1(s) + ib_2(s)\}_{s \geqslant 0}$ on the complex plane $\mathbb{C} \cong \mathbb{R} \times \mathbb{R}$. More precisely, the coherent representation of $e^{-t\mathcal{H}}$ may, for all $z, z' \in \mathbb{C}$ and $t > 0$, be written as

$$\langle z|e^{-t\mathcal{H}}|z'\rangle = \lim_{\nu\to\infty} \int d\mu^{(\nu)}_{z,0;z',t}(b) \exp\left\{ 4(j+1)\nu \int_0^t \frac{ds}{(1+|b(s)|^2)^2} \right\} \times \exp\left\{ (j+1) \int_0^t ds \frac{\dot{b}(s)b^*(s) - \dot{b}^*(s)b(s)}{1+|b(s)|^2} - \int_0^t ds\, h(b(s)) \right\}. \tag{5}$$

Here for given z, $z' \in \mathbb{C}$, $t > 0$, and $\nu > 0$ the path integration is defined by

$$\int d\mu^{(\nu)}_{z,0;z',t}(b)(\cdot) := \frac{1}{4\pi t\nu} e^{-|z-z'|^2/4t\nu} \mathbb{E}(\cdot), \tag{6}$$

where $\mathbb{E}(\cdot)$ indicates the probabilistic expectation with respect to the *two-dimensional Brownian bridge*, with diffusion constant ν starting in $z = b(0)$ and arriving at $z' = b(t)$ a time t later.[3,6,37–39] As a Gaussian stochastic process with continuous paths on $\mathbb{C} \cong \mathbb{R} \times \mathbb{R}$ the Brownian bridge, in its turn, is uniquely determined by its mean,

$$\mathbb{E}(b(s)) = z + (z'-z)\frac{s}{t}, \quad s \in [0,t], \tag{7}$$

and covariances,

$$\mathbb{E}(b^*(r)b(s)) - \mathbb{E}(b^*(r))\mathbb{E}(b(s)) = 4\nu\left(\min\{r,s\} - \frac{rs}{t} \right), \tag{8}$$

$$\mathbb{E}(b(r)b(s)) - \mathbb{E}(b(r))\mathbb{E}(b(s)) = 0, \quad r,s \in [0,t]. \tag{9}$$

The second integral in the exponent on the right-hand side of (5) is a purely imaginary stochastic (line) integral,[37–39] which is understood in the sense of Fisk and Stratonovich and to which one is therefore allowed to apply the rules of ordinary calculus,[40] although the time derivative $\dot{b}$ does not exist.

Several comments apply.

(i) By the Itô formula[3,37–39] it can be seen that the stochastic integral in (5) may equally well be interpreted as a stochastic integral in the sense of Itô. Moreover, using the Itô formula in a different way, the sum of this integral and the first (Lebesgue) integral in the exponent of the right-hand side of (5) can be converted[41] according to

$$4\nu \int_0^t \frac{ds}{(1+|b(s)|^2)^2} + \int_0^t ds \frac{\dot{b}(s)b^*(s) - \dot{b}^*(s)b(s)}{1+|b(s)|^2} = \ln\left(\frac{1+|b(t)|^2}{1+|b(0)|^2} \right) - 2\int_0^t \frac{db^*(s)b(s)}{1+|b(s)|^2}. \tag{10}$$

Here the complex stochastic integral $\int_0^t db^*(s)b(s)/[1+|b(s)|^2]$ has to be understood in the sense of Itô. It contains the only ν-dependence of the right-hand side. By using (10) in the path integrand in (5), the logarithmic term results in the prefactor $[(1+|z'|^2)/(1+|z|^2)]^{j+1} = g(z)/g(z')$.

(ii) The stochastic integral in (5) is of kinematical origin and reflects the symplectic structure, which renders the complex plane a phase space for the so-called classical spin;[42,43,31] also see the concluding remarks.

(iii) If one wants to use (5) to express the trace $\int_{\mathbb{C}} d^2z \langle z|e^{-t\mathcal{H}}|z\rangle$ of $e^{-t\mathcal{H}}$ as a path integral, one should resist the temptation to interchange the integration with respect to z with the ultradiffusive limit $\nu \to \infty$, because the resulting prelimit expression would be infinite.

(iv) Instead of taking the ultradiffusive limit, one may perform the regularization also by a *long-time limit*, in the sense that

$$\langle z|e^{-t\mathcal{H}}|z'\rangle=\lim_{u\to\infty}\int d\mu^{(\nu)}_{z,0;z',u}(b)\exp\left\{4(j+1)\nu\int_0^u\frac{ds}{(1+|b(s)|^2)^2}\right\}$$
$$\times\exp\left\{(j+1)\int_0^u ds\,\frac{\dot b(s)b^*(s)-\dot b^*(s)b(s)}{1+|b(s)|^2}-\frac{t}{u}\int_0^u ds\,h(b(s))\right\}. \tag{11}$$

This formula can be deduced from (5) by suitably scaling the Brownian bridge, holds for all $\nu>0$, and, in contrast to (5), makes sense as it stands even for $t\leqslant 0$, hence for all $t\in\mathbb{R}$. One should notice that the time-parameter set of the Brownian bridge used in (11) is the closed interval $[0,u]$ and not $[0,t]$.

(v) Replacing h by ih in (5) or (11) yields analogous expressions for the coherent representation of the (unitary) *spin time-evolution* operator $e^{-it\mathcal{H}}$. A rigorous justification relies on the boundedness and continuity of h and requires extending the subsequent proof by showing analyticity of both sides of (5) in a coupling parameter $\lambda\in\mathbb{C}$ multiplying h. The left-hand side and the prelimit expression in (5) are easily seen to be analytic in λ. Analyticity in λ in the limit $\nu\to\infty$ is then proved with the help of an equation analogous to (29) and uniform convergence in $\nu>2\nu_0>0$ of the perturbation series in λ of the relevant operator and functions there.

(vi) The flat-space Wiener-regularized path-integral expression (5) for the spin semigroup is an alternative to a result first given and proved in Ref. 12. There the authors integrate over Brownian-motion paths on the unit-sphere in the three-dimensional Euclidean space $\mathbb{R}^3$ to obtain the coherent representation of $e^{-it\mathcal{H}}$. Unlike in Ref. 12, the regularizing path measure $d\mu^{(\nu)}_{z,0;z',t}(b)\exp\{4(j+1)\nu\int_0^t ds(1+|b(s)|^2)^{-2}\}$ used in (5) is not invariant under the full special unitary group SU(2) when the latter is realized by suitable Möbius transformations on the (extended) complex plane. Yet in the limit $\nu\to\infty$ all symmetries of a given spin Hamiltonian are restored. Contrary to what one might expect, Eq. (5) cannot be obtained from the corresponding result in Ref. 12 merely by stereographically projecting the paths from the sphere onto the (extended) plane. Nevertheless, the proof given in the next section shows that the key ideas behind both constructions are the same; also see the concluding remarks.

(vii) So far we have considered a fixed spin quantum number j. In order to make contact with the Wiener-regularized path-integral expression associated with a canonical degree of freedom, also proved in Ref. 12, one has to contract[44,45] the algebra of SU(2) to the Heisenberg–Weyl algebra by taking the *high-spin limit* $j\to\infty$. More explicitly, in the given (polynomial) spin Hamiltonian $\mathcal{H}$ on $\mathbb{C}^{2j+1}$, one has to replace $\mathcal{J}_+$, $\mathcal{J}_-$, and $\mathcal{J}_3$ by $\mathcal{J}_+/\sqrt{2j}$, $\mathcal{J}_-/\sqrt{2j}$, and $\mathcal{J}_3+j\mathbf{1}$, respectively. If $\mathcal{H}_j=\int_{\mathbb{C}}d^2z\,h_j(z)|z\rangle\langle z|$ denotes the resulting operator, one then finds the relation

$$\lim_{j\to\infty}\frac{\pi}{2j}\langle z/\sqrt{2j}|e^{-t\mathcal{H}_j}|z'/\sqrt{2j}\rangle=\langle\langle z|e^{-t\mathsf{H}}|z'\rangle\rangle, \tag{12}$$

where $|z\rangle\rangle\in L^2(\mathbb{R})$ is a normalized canonical coherent vector[30–32] and the Hamiltonian H on $L^2(\mathbb{R})$, the Hilbert space of Lebesgue square-integrable complex-valued functions on the real line $\mathbb{R}$, is defined by

$$\mathsf{H}:=\int_{\mathbb{C}}\frac{d^2z}{\pi}\mathsf{h}(z)|z\rangle\rangle\langle\langle z|,\quad\text{with}\quad \mathsf{h}(z):=\lim_{j\to\infty}h_j(z/\sqrt{2j}). \tag{13}$$

By using (5) for the prelimit expression in (12), suitably rescaling the Brownian bridge, and interchanging the order of the limits $j\to\infty$ and $\nu\to\infty$, one arrives at the path-integral formula

$$\langle\langle z|e^{-t\mathsf{H}}|z'\rangle\rangle=\pi\lim_{\nu\to\infty}e^{2t\nu}\int d\mu^{(\nu)}_{z,0;z',t}(b)\exp\left\{\frac{1}{2}\int_0^t ds[\dot b(s)b^*(s)-\dot b^*(s)b(s)]\right\}$$
$$\times\exp\left\{-\int_0^t ds\,\mathsf{h}(b(s))\right\}, \tag{14}$$

in agreement with Eq. (1.3) in Ref. 12; also see Refs. 46 and 35. Formula (14) can be shown to hold not only for the polynomial Hamiltonians $\mathcal{H}$ resulting from the contraction, but for a wider class of operators whose conditions are stated in Theorem 2.4 of Ref. 12.

(viii) With regard to some of the symbolic path-integral expressions for spin systems frequently encountered in the literature, see, for example, Refs. 10, 11, and 23–28, it might be illuminating to recognize certain formal similarities between these expressions and the above result (5). While the kinematical and dynamical terms in the exponents of all the corresponding path integrands look essentially the same, only the above result is based on a genuine path measure, namely, $d\mu^{(\nu)}_{z,0;z',t}(b)\exp\{4(j+1)\nu\int_0^t ds(1+|b(s)|^2)^{-2}\}$, but requires taking the limit $\nu\to\infty$. Here, the Wiener type of measure $d\mu^{(\nu)}_{z,0;z',t}(b)$ is often symbolically written as $\delta^2 b\ \delta(b(0)-z)\,\delta(b(t)-z')\exp\{-(1/4\nu)\int_0^t ds|\dot{b}(s)|^2\}$, or similarly. In any case, the necessity to regularize by some ultradiffusive limit was observed several times also in nonrigorous works.[10,17–19,27]

III. PROOF

The proof of (5) consists of three major steps, adapting key ideas of Ref. 12. First, the spin Hilbert space $\mathbb{C}^{2j+1}$ is embedded into $L^2(\mathbb{C})$, the Hilbert space of Lebesgue square-integrable complex-valued functions on $\mathbb{C}$. Next, it is identified with the $(2j+1)$-dimensional ground-state eigenspace of a suitable Schrödinger operator R acting on $L^2(\mathbb{C})$. Then the spin semigroup, now realized on $L^2(\mathbb{C})$, is shown to be the limit $\nu\to\infty$ of a Schrödinger semigroup generated by a suitably perturbed νR. Rewriting this Schrödinger semigroup with the help of the standard Feynman–Kac–Itô path-integral formula finally gives (5).

A. The embedding of the spin Hilbert space

The embedding of the spin Hilbert space $\mathbb{C}^{2j+1}$ into the infinite-dimensional Hilbert space $L^2(\mathbb{C})$, equipped with the standard scalar product $(\varphi|\psi):=\int_{\mathbb{C}}d^2z\ \varphi^*(z)\psi(z)$, is accomplished by interpreting the coherent representation as a linear isometric mapping,

$$I:\ \mathbb{C}^{2j+1}\to L^2(\mathbb{C}),\quad |\psi\rangle\mapsto\psi, \tag{15}$$

where the function ψ on $\mathbb{C}\cong\mathbb{R}\times\mathbb{R}$ is defined by its values $\psi(z):=\langle z|\psi\rangle$.

The (Hilbert) adjoint $I^\dagger$ of I explicitly reads as

$$I^\dagger:\ L^2(\mathbb{C})\to\mathbb{C}^{2j+1},\quad \varphi\mapsto\int_{\mathbb{C}}d^2z\ \varphi(z)|z\rangle, \tag{16}$$

and the isometric property is simply stated as $I^\dagger I=\mathbf{1}$. The orthogonal projection from $L^2(\mathbb{C})$ onto $I(\mathbb{C}^{2j+1})$ is the operator $II^\dagger=:E_0$.

Every operator $\mathcal{B}$ on $\mathbb{C}^{2j+1}$ can be realized by the unitary equivalent $I\mathcal{B}I^\dagger$ on $E_0(L^2(\mathbb{C}))=I(\mathbb{C}^{2j+1})$, which trivially extends to the whole of $L^2(\mathbb{C})$. In particular, it follows from (4) that

$$I\mathcal{H}I^\dagger=E_0HE_0, \tag{17}$$

where H is the bounded multiplication operator on $L^2(\mathbb{C})$ defined by the function h, that is, $(H\varphi)(z):=h(z)\varphi(z)$ for all $\varphi\in L^2(\mathbb{C})$. Furthermore, the embedded operator $I\mathcal{H}I^\dagger$ possesses a continuous integral kernel $I\mathcal{H}I^\dagger(z,z')$ (also known as its position representation) given by the coherent representation of $\mathcal{H}$, that is,

$$I\mathcal{H}I^\dagger(z,z')=\langle z|\mathcal{H}|z'\rangle. \tag{18}$$

Using (17), one can now verify the identify $Ie^{-t\mathcal{H}}I^\dagger=E_0e^{-tE_0HE_0}$ to all orders in t, which, analogous to (18), shows that $E_0e^{-tE_0HE_0}$ has a continuous integral kernel given by the equation

$$E_0e^{-tE_0HE_0}(z,z')=\langle z|e^{-t\mathcal{H}}|z'\rangle. \tag{19}$$

B. A Schrödinger operator and its ground-state eigenspace

Consider on $L^2(\mathbb{C})$ the ''magnetic'' Schrödinger operator,

$$R:=(i\partial_1+A_1)^2+(i\partial_2+A_2)^2+V, \tag{20}$$

with the partial differential operators $\partial_1:=\partial/\partial z_1$, $\partial_2:=\partial/\partial z_2$ and the vector and scalar potentials $\binom{A_1}{A_2}$ and V acting as multiplication operators defined by the bounded and continuous functions,

$$\begin{pmatrix} a_1(z) \\ a_2(z) \end{pmatrix} := \begin{pmatrix} -\partial_2 \\ \partial_1 \end{pmatrix} \ln g(z) = \frac{2(j+1)}{1+|z|^2}\begin{pmatrix} z_2 \\ -z_1 \end{pmatrix}, \tag{21}$$

$$v(z):=\partial_1 a_2(z)\text{-}\partial_2 a_1(z) = -\frac{4(j+1)}{(1+|z|^2)^2}. \tag{22}$$

The self-adjoint operator R is tailored such that its ground-state eigenspace is identical to $E_0(L^2(\mathbb{C}))$ and the corresponding eigenvalue vanishes. In essence, this follows from a result of Aharonov and Casher[47] on zero-energy eigenstates. Since the proof is quite short, we will give it, thereby closely following the presentation in Ref. 48. Factorized like $R=D^\dagger D$, where $D:=i\partial_1+\partial_2+A_1-iA_2$, the positivity of R becomes manifest. Its null space consists of all those functions ψ in $L^2(\mathbb{C})$ with $D\psi=0$. The general solution of this differential equation is a product $\psi=g\phi$, where ϕ is any function analytic in z^*, that is, $(\partial_1-i\partial_2)\phi=0$. Due to (3), square integrability then requires ϕ to be any polynomial in z^* of maximal degree $2j$, which proves that the ground-state eigenspace of R and the subspace $E_0(L^2(\mathbb{C}))=I(\mathbb{C}^{2j+1})$ are identical.

Two remarks are in order.

(i) The spectrum of R coincides with the positive half-line, as can be inferred from Theorem 6.1 in Ref. 48. Following arguments as in the proof of Theorem 6.2 in Ref. 48, one sees that zero is the only eigenvalue. Therefore the nature of the spectrum and the ground-state eigenfunctions are explicitly known. However, we are not aware of explicit results on generalized eigenfunctions corresponding to strictly positive spectral values.

(ii) Employing the spectral theorem, one proves that the semigroup generated by νR converges strongly to the ground-state projection E_0, in the sense that

$$\lim_{\nu\to\infty}\|e^{-t\nu R}\varphi - E_0\varphi\| = 0, \quad \text{for all} \quad \varphi\in L^2(\mathbb{C}) \quad \text{and} \quad t>0, \tag{23}$$

where the norm $\|\cdot\|:=(\cdot|\cdot)^{1/2}$ corresponds to the standard scalar product on $L^2(\mathbb{C})$.

C. The spin semigroup as the limit of a Schrödinger semigroup

With the material gathered in Secs. III A and III B we can isolate the central reason for the validity of the main result (5) of the present paper. The point is that the spin semigroup, now realized on $L^2(\mathbb{C})$, can be understood as the limit $\nu\to\infty$ of the Schrödinger semigroup generated by $\nu R+H$. More precisely, we will show that the continuous integral kernel given in (19) is the pointwise limit

$$E_0 e^{-tE_0HE_0}(z,z') = \lim_{\nu\to\infty} e^{-t(\nu R+H)}(z,z'), \quad \text{for all} \quad z,z'\in\mathbb{C} \quad \text{and} \quad t>0, \tag{24}$$

where the prelimit expression is the continuous integral kernel of $\exp\{-t(\nu R+H)\}$. By expressing this integral kernel in terms of the *Feynman–Kac–Itô formula*[49,50] (observing $\partial_1 a_1+\partial_2 a_2=0$)

$$e^{-t(\nu R+H)}(z,z')=\int d\mu^{(\nu)}_{z,0;z',t}(b)\exp\left\{(j+1)\int_0^t ds\,\frac{\dot b(s)b^*(s)-\dot b^*(s)b(s)}{1+|b(s)|^2}\right\}$$
$$\times\exp\left\{4(j+1)\nu\int_0^t\frac{ds}{(1+|b(s)|^2)^2}-\int_0^t ds\,h(b(s))\right\},\tag{25}$$

the right-hand sides of (24) and (5) are seen to coincide.

The proof of (24) makes essential use of the semigroup property of $e^{-t(\nu R+H)}$. Throughout the proof we fix $t>0$ and pick some reference diffusion constant $\nu_0>0$. As a starting point we define

$$\eta^{(\lambda)}_w(z):=e^{-t(\nu_0 R+\lambda H)}(z,w),\tag{26}$$

for all $\lambda\in\mathbb{R}$ and $w,z\in\mathbb{C}$. We assert that the function $\eta^{(\lambda)}_w:z\mapsto\eta^{(\lambda)}_w(z)$ is continuous, bounded, and lies in $L^2(\mathbb{C})$. The continuity follows from that of the integral kernel in (25). Boundedness and square integrability result from the inequality

$$|\eta^{(\lambda)}_w(z)|\leqslant e^{4(j+1)t\nu_0}e^{t|\lambda|\|h\|_\infty}(4\pi t\nu_0)^{-1}e^{-|z-w|^2/4t\nu_0},\tag{27}$$

where $\|h\|_\infty:=\sup_{z\in\mathbb{C}}|h(z)|<\infty$ denotes the supremum norm of h. This inequality, in turn, is found by estimating the path integral in (25). We also state that the mappings $w\mapsto\eta^{(\lambda)}_w$ and $\lambda\mapsto\eta^{(\lambda)}_w$ are strongly continuous. The first statement holds because of $(\eta^{(\lambda)}_w|\eta^{(\lambda)}_{w'})=e^{-2t(\nu_0 R+\lambda H)}(w,w')$ and the continuity of the integral kernel. The second one is a consequence of the inequality

$$\|\eta^{(\lambda)}_w-\eta^{(\lambda')}_w\|\leqslant\sqrt{\frac{t}{8\pi\nu_0}}|\lambda-\lambda'|\|h\|_\infty e^{4(j+1)t\nu_0}e^{t\max\{|\lambda|,|\lambda'|\}\|h\|_\infty},\tag{28}$$

which is derived by estimating the difference of two path integrals of type (25) using the elementary inequality $|e^x-e^y|\leqslant|x-y|e^{\max\{x,y\}}$, for $x,y\in\mathbb{R}$.

The following two steps of the proof are based on writing the integral kernel for $\nu>2\nu_0$ as a scalar product,

$$e^{-t(\nu R+H)}(z,z')=(\eta^{(\nu_0/\nu)}_z|e^{-t(\nu-2\nu_0)(R+H/\nu)}\eta^{(\nu_0/\nu)}_{z'}).\tag{29}$$

In the first step, we claim that

$$\lim_{\nu\to\infty}(\eta^{(\nu_0/\nu)}_z|e^{-t(\nu-2\nu_0)(R+H/\nu)}\eta^{(\nu_0/\nu)}_{z'})=(\eta^{(0)}_z|E_0e^{-tE_0HE_0}\eta^{(0)}_{z'}),\tag{30}$$

for all $z,z'\in\mathbb{C}$.

Due to the strong continuity of $\lambda\mapsto\eta^{(\lambda)}_w$, the boundedness of $e^{-t(\nu-2\nu_0)(R+H/\nu)}$, which is uniform in ν, and the continuity of the scalar product $(\cdot|\cdot)$, it suffices to show that

$$\lim_{\nu\to\infty}\|e^{-t(\nu-2\nu_0)(R+H/\nu)}\varphi-E_0e^{-tE_0HE_0}\varphi\|=0,\quad\text{for all}\quad\varphi\in L^2(\mathbb{C}).\tag{31}$$

To prove this strong operator convergence we employ the Duhamel–Dyson–Phillips perturbation expansion,

$$e^{-t(\nu-2\nu_0)(R+H/\nu)}\varphi=e^{-t(\nu-2\nu_0)R}\varphi+\sum_{n=1}^\infty\left(\frac{2\nu_0-\nu}{\nu}\right)^n\int_0^t ds_n\cdots\int_0^{s_3}ds_2\int_0^{s_2}ds_1\,e^{-(t-s_n)(\nu-2\nu_0)R}$$
$$\times H\times\cdots\times e^{-(s_2-s_1)(\nu-2\nu_0)R}He^{-s_1(\nu-2\nu_0)R}\varphi,\tag{32}$$

which converges uniformly in $\nu \in]2\nu_0,\infty[$ with respect to the norm on $L^2(\mathbb{C})$. This holds because the norm of the series is dominated by the exponential series $\Sigma_{n=0}^{\infty}(t^n/n!)\|h\|_\infty^n\|\varphi\|$, independent of ν. Thus, the limit $\nu\to\infty$ can be interchanged with the summation and, using (23) and the dominated-convergence theorem, we obtain the expansion of $E_0 e^{-tE_0HE_0}\varphi$.

In the second and final step we claim that the right-hand side of (30) is already the desired integral kernel, that is,

$$(\eta_z^{(0)}|E_0 e^{-tE_0HE_0}\eta_{z'}^{(0)}) = E_0 e^{-tE_0HE_0}(z,z'), \quad \text{for all } z,z'\in\mathbb{C}. \tag{33}$$

This is verified by checking that the mapping $(z,z')\mapsto(\eta_z^{(0)}|E_0e^{-tE_0HE_0}\eta_{z'}^{(0)})$ constitutes an integral kernel of $E_0e^{-tE_0HE_0}$ and is, in fact, continuous. The former is true since $e^{-t\nu_0R}E_0=E_0$. The latter holds because the mapping $w\mapsto\eta_w^{(0)}$ is strongly continuous, the operator $E_0e^{-tE_0HE_0}$ is bounded, and the scalar product $(\cdot|\cdot)$ is continuous.

IV. CONCLUDING REMARKS

We conclude the paper with six remarks.

(i) As already mentioned in Sec. II, the main result (5) cannot be obtained from a result in Ref. 12 merely by stereographically projecting the Brownian paths from the two-sphere S^2 onto the (extended) Euclidean plane $\mathbb{R}^2$. The reason can be traced back to the different operators, or equivalently path measures, used for regularization. The stereographic projection corresponds to reexpressing the differential operator on $L^2(S^2)$ used by the authors of Ref. 12 in flat Cartesian coordinates. The resulting operator is not of the standard Schrödinger form, acts on a weighted Hilbert space, and is not related to planar Brownian motion.

(ii) In contrast to Ref. 12 the regulatizing operator R used in the proof of (24), and hence of (5), has no spectral gap above its ground-state eigenvalue. Accordingly, $e^{-t\nu R}$ only converges strongly, and not in operator norm, to the corresponding eigenprojection E_0 as $\nu\to\infty$. As a consequence, the foregoing proof of the pointwise convergence of integral kernels required a strategy different from that in Ref. 12.

(iii) From a fundamental point of view, it is gratifying that a spin system can be related to a limit of a well-defined integral over continuous Brownian-motion paths. From a practical point of view, it would be desirable to apply to (5) the well-established theory and computational possibilities associated with the flat-space Wiener measure,[3,39,51] in order to attack specific spin problems of physical interest. One such problem, which has been extensively discussed in the recent literature,[23–28] is to understand the nature of the saddle-point approximation for the evaluation of continuum path integrals connected with simple spin Hamiltonians. Looking at Table I and the resulting j-dependence of the path integrand in (5), this approximation is expected to be the more reliable the larger the spin quantum number is. Moreover, for Hamiltonians $\mathcal{H}$ linear in the spin operators, the saddle-point approximation is believed[23–27] to give the (explicitly known) exact result already for given finite j. In this context, when dealing with symbolic continuum path integrals one has to overcome the so-called overspecification problem due to missing regularizing terms in the action functionals of those path integrals.[10,27] Rigorous continuum path integrals as used in (5) do not suffer from this problem by their very construction. Of course, the details for the saddle-point approximation of the Wiener type of path integral in the ultradiffusive limit still have to be worked out.

(iv) In Refs. 52 and 53, the ground-state eigenspace of a charged point mass under the influence of a certain magnetic field on an even-dimensional Riemannian manifold is studied, thereby extending the Aharonov–Casher theorem.[47,48] This result lies at the heart of the quantization procedure proposed in Refs. 53–55. A quantum system is hereby represented on the ground-state eigenspace of such a generalized Landau Hamiltonian on the Hilbert space of functions over its classical phase space. The symplectic structure of the latter determines the magnetic field. In this sense, Eq. (5) read from right to left can be viewed as a *quantization prescription* for a classical spin system.

In this context it is worth mentioning that the path integral in (5) is well-defined for all values of j taken from the positive half-line. Even more, in the limit $\nu\to\infty$ it manages to single out the set of allowed spin quantum numbers, {0,1/2,1,3/2,...}, from the ''classical continuum $[0,\infty[$.

More precisely, for a given bounded and continuous h: $\mathbb{C}\to\mathbb{C}$ and $j\in[0,\infty[$ we assert that the right-hand side of (5) is equal to $\langle\psi(z)|e^{-t\mathcal{H}_\psi}|\psi(z')\rangle$. Here the set of vectors,

$$|\psi(z)\rangle := g(z)\sum_{n=0}^{2(j)} \sqrt{\binom{2j}{n}}z^n|\psi_n\rangle, \quad z\in\mathbb{C}, \tag{34}$$

is unity-resolving in $\mathbb{C}^{2(j)+1}$, where (j) denotes the smallest integer or half-integer equal to or larger than j, and $\{|\psi_n\rangle\}$ is a fixed but arbitrary orthonormal basis in $\mathbb{C}^{2(j)+1}$. The binomial coefficient can be defined recursively by $\binom{2j}{0}:=1$ and $\binom{2j}{n+1}:=[(2j-n)/(n+1)]\binom{2j}{n}$, and $g(z)$ is defined by (3) as it stands for general $j\in[0,\infty[$. Finally, $\mathcal{H}_\psi$ is an operator on $\mathbb{C}^{2(j)+1}$ associated to the given h by the definition

$$\mathcal{H}_\psi := \int_{\mathbb{C}} d^2z\, h(z)|\psi(z)\rangle\langle\psi(z)|. \tag{35}$$

This association can be viewed as a quantization, which maps the pair (j,h) to the pair $((j),\mathcal{H}_\psi)$ with $\mathcal{H}_\psi$ being interpreted as the Hamiltonian of a spin with quantum number (j). While $\mathcal{H}_\psi$ in general depends on the chosen basis $\{|\psi_n\rangle\}$, the expression $\langle\psi(z)|e^{-t\mathcal{H}_\psi}|\psi(z')\rangle$ does not because of unitary invariance.

For the proof of the above assertion we remark that the latter is identical to (5) in the case $j=(j)$, because then $|\psi(z)\rangle=|z\rangle$ when choosing $|\psi_n\rangle=|j,n-j\rangle$, the usual orthonormal eigenbasis of $\mathcal{J}_3$. In the case $j<(j)$, the proof follows from (25), equations analogous to (24) and (19), and the Aharonov-Casher theorem, which in our setting states that the ground-state eigenspace of the ''magnetic'' Schödinger operator R [stemming from g, confer (20)–(22)] has a dimension equal to the largest integer strictly smaller than $|\int_{\mathbb{C}}d^2z v(z)|/2\pi=2j+2$ and is spanned by the set of orthonormal functions $z\mapsto\langle\psi(z)|\psi_n\rangle$, $n=0,1,\ldots,2(j)$.

(v) It is straightforward to generalize formula (5) to systems where the Hamiltonian $\mathcal{H}$ depends explicitly on time and/or several (coupled) spins. The formula in the latter case, like its older ''spherical relative'' in Ref. 12, may then serve as a rigorous starting point for the derivation of effective field theories, which aim to describe the low-energy excitations of quantum lattice models for magnetism. Confer, for example, Refs. 17, 56, 57, and references therein.

(vi) Following the reasoning of the present paper it should also be straightforward to derive flat-space Wiener-regularized path integrals also for physical systems with degrees of freedom that are neither of the canonical nor of the spin type.

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